Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

ON THE CONFORMAL REPRESENTATION OF CERTAIN ISOSCELES TRIANGLES UPON THE UPPER HALF PLANE

By L. Wayland Dowling

1. Introduction. The problem of conformally representing a polygon whose sides are right lines or arcs of circles, upon the upper half of the z-plane, was completely solved more than thirty years ago by Christoffel and Schwarz.*

If the angles of a rectilinear triangle, situated in the w-plane, are $\alpha\pi$, $\beta\pi$, and $\gamma\pi$, then the representation is effected by the formula

$$C_1 w + C_2 = \int \frac{dz}{(z-a)^{1-\alpha}(z-b)^{1-\beta}(z-c)^{1-\gamma}},$$

where the constants a, b, and c, are the values of z (real) corresponding to the vertices of the triangle. The constants C_1 and C_2 depend upon the position of the triangle in the w-plane.

In the following pages this formula is applied to certain plane triangles, in particular, isosceles triangles, and the discussion is limited to the simplest cases, viz: triangles whose angles are rational proper fractions of 180°. Obviously any such triangle can be formed from a regular polygon by joining three of its vertices, provided the number of sides of such a polygon is a common multiple of the denominators of α, β, and γ.

It is assumed, for convenience, that the constants a, b, and c, have the values -1, $+1$, and 0 respectively, and that the lowest common denominator of the fractions α, β, and γ, is n, so that the polygon of *least* number of sides from which the triangle can be formed is an n-gon. Furthermore, putting $1-\alpha=\lambda_1/n$, $1-\beta=\lambda_2/n$, $1-\gamma=\lambda_3/n$, the Christoffel-Schwarz formula becomes

$$C_1 w + C_2 = \int \frac{dz}{\sqrt[n]{(z+1)^{\lambda_1}(z-1)^{\lambda_2} z^{\lambda_3}}}. \tag{1}$$

In the formula thus written, λ_1, λ_2, and λ_3 are integers satisfying the conditions

$$\Sigma\lambda_i = 2n;\ 1 < \lambda_i < n;\ (i = 1, 2, 3). \tag{2}$$

---

* Christoffel: *Ann. di. mat.*, ser. 2, vol. 1 (1867), p. 97.
Schwarz: *Crelle*, vol. 70 (1869), p. 117; *Werke*, vol. 2, p. 80.

Every set of three integers satisfying these conditions may be taken to represent a triangle; the symbol $[\lambda_1, \lambda_2, \lambda_3]$ designating the triangle whose angles are

$$\left(1-\frac{\lambda_1}{n}\right)\pi,\quad \left(1-\frac{\lambda_2}{n}\right)\pi,\quad \text{and}\quad \left(1-\frac{\lambda_3}{n}\right)\pi.$$

The representation is obviously not unique, for any given triangle will be represented by an infinity of such sets of numbers just as it can be formed from an infinity of regular polygons by joining vertices. Any given set of numbers, however, satisfying conditions (2), can represent but one triangle.

The integral in (1) is an Abelian integral of the first kind existing upon the n-leaved Riemann surface

$$(3)\qquad u^n = (z+1)^{\lambda_1}(z-1)^{\lambda_2}z^{\lambda_3}.$$

This surface has winding points at $z = -1, +1$, and 0, and nowhere else. If $p_i = [n ; \lambda\ ]$, *i. e.*, the highest common factor of n and λ_i, then, at the winding point for which λ_i is the exponent, the n leaves are grouped into p_i groups, each group consisting of n/p_i leaves winding together. Hence the deficiency of the surface is

$$(4)\qquad p = \frac{\Sigma p_i\left(\dfrac{n}{p_i}-1\right)}{2} - n + 1 = \frac{n+2-\Sigma p_i}{2}.$$

Besides the integral (1) there are $p-1$ other integrals of the first kind existing upon the same surface and linearly independent of (1).

2. The complete set of integrals of the first kind. The first question will be to find out the meaning which such a set of independent integrals has in the problem of the conformal representation of the given triangle $[\lambda_1, \lambda_2, \lambda_3]$. The theory of Abelian integrals teaches that, for proper values of the integers k, s_1, s_2, s_3, the integral

$$(5)\qquad I_k = \int \frac{(z+1)^{s_1}(z-1)^{s_2}z^{s_3}}{u^k}\,dz \equiv \int \frac{dz}{\sqrt{(z+1)^{\mu_1}(z-1)^{\mu_2}z^{\mu_3}}}$$

will be an integral of the first kind upon the same surface (3) and will be independent of (1). The necessary and sufficient conditions to be imposed upon the μ's are:

$$(6)\qquad k\lambda_i \equiv \mu_i \pmod{n},\ \Sigma\mu_i = 2n,\ 1 < \mu_i < n;\ (i = 1, 2, 3).$$

The first of these conditions states that the μ's are derived from the *given* λ's as is indicated in (5). The conditions $\Sigma\mu_i = 2n$ and $1 < \mu_i < n$ insure the finiteness of the integral over the entire Riemann surface.

It is necessary to show, first of all, that the integrals I_k form a *complete set* of linearly independent integrals when the conditions (6) are satisfied. That the integrals I_k are linearly independent follows from the theory as above noted.

There are obviously not more than $n - 1$ different sets of μ's satisfying the conditions $k\lambda_i \equiv \mu_i \pmod{n}$ and $1 < \mu_i < n$. In fact if n is prime to λ_1, λ_2, and λ_3 there will be *just* $n - 1$ sets of μ's satisfying the above congruences and the conditions $1 \leqq \mu_i < n$. But if n is not prime to each of the λ's and if, as before, p_i stands for $[n; \lambda_i]$, then the theory of linear congruences shows that there are $p_i - 1$ values of k other than zero for which $k\lambda_i \equiv 0 \pmod{n}$; *i. e.*, there will be $p_i - 1$ of the μ_i's equal to zero and these must be thrown out as not satisfying the conditions $1 < \mu_i < n$. Hence there cannot be more than $n - 1 - \Sigma(p_i - 1)$ sets of μ's satisfying the first and third of conditions (6). Of these sets, however, not all will satisfy the remaining condition, $\Sigma\mu_i = 2n$. For, suppose k_1 is a multiplier giving rise to a set of μ's satisfying all the conditions (6), then the multiplier $n - k_1$ will give rise to a set of μ's not satisfying the condition $\Sigma\mu_i = 2n$, since $(n - k_1)\lambda_i$ is obviously congruent to $n - \mu_i \pmod{n}$ if $k_1\lambda_i \equiv \mu_i \pmod{n}$. But $\Sigma(n - \mu_i) = n$. Hence there cannot be more than

$$\frac{(n-1) - \Sigma(p_i - 1)}{2} = \frac{n + 2 - \Sigma p_i}{2} = p$$

sets of μ's satisfying all of the conditions (6). To see that there is exactly this number notice that for every set of μ's satisfying the congruences $k\lambda_i \equiv \mu_i \pmod{n}$ we must have $\Sigma\mu_i \equiv 0 \pmod{n}$ because $\Sigma\lambda_i = 2n$. Imposing the conditions $1 < \mu_i < n$ we see that $\Sigma\mu_i$ must be either n or $2n$, and for every value of k that makes $\Sigma\mu_i = 2n$ there is another, namely $n - k$, that makes $\Sigma\mu_i = n$, and *vice versa.* Therefore there are exactly p sets of integers μ satisfying conditions (6) and hence exactly p of the integrals I_k; *i. e.*, the I_k's form a *complete* set of linearly independent integrals upon the Riemann surface (3).

As an example, consider the triangle $[7, 8, 11]$ whose angles are $\frac{6}{13}\pi$, $\frac{5}{13}\pi$, $\frac{2}{13}\pi$. It is represented conformally upon the upper half plane by the integral

$$I_1 = \int \frac{dz}{\sqrt[13]{(z+1)^7\,(z-1)^8 z^{11}}}\,.$$

The integrals I_1, I_3, I_7, I_8, I_9, and I_{11} form a complete set of linearly independent integrals upon the surface $u^{13} = (z+1)^7\,(z-1)^8\,z^{11}$, any one of which may be written down at once, for instance:

$$I_8 = \int \frac{(z+1)^4\,(z-1)^4 z^6\,dz}{u^8}\,.$$

Now the μ's corresponding to any one of the I_k's must satisfy precisely the conditions (2) imposed upon the λ's and hence it follows that each integral I_k represents some triangle conformally upon the upper half-plane; moreover, this triangle is formed from exactly the same regular polygon as was the original triangle. Therefore:

The complete set of linearly independent integrals of the first kind, I_k, *existing upon the Riemann surface* $u^n = (z+1)^{\lambda_1}\,(z-1)^{\lambda_2} z^{\lambda_3}$ *represents conformally upon the upper half-plane a set of triangles each of which can be formed from the same regular n-gon by joining three of its vertices in the proper manner.*

This theorem sets up an intimate relation between a particular set of linearly independent integrals of the first kind existing upon the Riemann surface (3) and a particular set of triangles formed from a regular n-gon. It does not assert, however, that the triangles will necessarily be all distinct. In fact, the same triangle will, in general, be represented conformally upon the upper half-plane by different integrals if its vertices be permuted. In the examples quoted above we have the correspondences

$$\begin{array}{llll} I_1 \cdots [\;7,\; 8,\; 11], & & I_7 \cdots [10,\; 4,\; 12], \\ I_3 \;.\;.\;.\; [\;8,\; 11,\; 7], & & I_8 \;.\;.\;.\; [\;4,\; 12,\; 10], \\ I_9 \cdots [11,\; 7,\; 8], & & I_{11} \cdots [12,\; 10,\; 4]. \end{array}$$

Here, I_1, I_3, and I_9 represent the same triangle with its vertices permuted cyclically, and the same is true of the integrals I_7, I_8, I_{11}.

3. The regular n-gon. A second question would be to discuss completely the triangles and corresponding integrals arising from any given n-gon. The general problem, however, is one of great complexity and does not seem to admit of concise statement or solution. Something may be said, however, and, first of all, the number of distinct triangles that can be formed from the

n-gon by joining vertices may be determined. For this purpose it is convenient to arrange the triangles in columns as follows:

$$\begin{array}{llll} [n-1, n-1, 2] & [n-2, n-2, 4] & \cdots & [n-k, n-k, 2k] \\ [n-1, n-2, 3] & [n-2, n-3, 5] & \cdots & [n-k, n-k-1, 2k+1] \\ \quad \cdots & \quad \cdots & & \quad \cdots \\ [n-1, n-r, r+1] & [n-2, n-s, s+2] & \cdots & [n-k, n-L, k+L] \\ \quad \cdots & \quad \cdots & & \quad \cdots \end{array}$$

The columns are to be continued until the λ_2, λ_3 of the last triangle in each are either equal or $\lambda_2 = \lambda_3 + 1$. In this way the scheme takes account of every possible distinct triangle without repetitions. Suppose the columns numbered from left to right in order and divided into two groups; group I containing the odd numbers and group II, the even numbers. The following statements are then easily verified:

1. The number of columns is always the greatest integer contained in $n/3$. For the number of columns is evidently equal to k, and $2k$ must be $\leqq$ to $n - k$, hence $k \leqq n/3$.
2. If $n \equiv 0, 1,$ or $2 \pmod 6$, the number of columns in each group will be equal; while if $n \equiv 3, 4,$ or $5 \pmod 6$, the number of columns in group I will exceed those in group II by unity.
3. The number of triangles in the columns of either group form an arithmetical progression whose common difference is -3. If n is even, the first term of this progression for group I is $\frac{1}{2}(n-2)$; for group II, $\frac{1}{2}(n-4)$. If n is odd the first term for group I is $\frac{1}{2}(n-1)$; for group II, $\frac{1}{2}(n-5)$.

These statements furnish the means for computing the number of triangles by a simple summation of the series involved. If ν represents the total number of triangles, the results may be concisely stated as follows:—

$$\text{If } n^2 \equiv 0 \pmod 6, \qquad \nu = \frac{n^2}{12};$$

$$\text{if } n^2 \equiv 1 \pmod 6, \qquad \nu = \frac{n^2 - 1}{12};$$

$$\text{if } n^2 \equiv 3 \pmod 6, \qquad \nu = \frac{n^2 + 3}{12};$$

$$\text{if } n^2 \equiv 4 \pmod 6, \qquad \nu = \frac{n^2 - 4}{12}.$$

Among the ν possible distinct triangles that can be formed from the given n-gon, there is one group which stands out prominently from the rest. This group contains all the possible distinct isosceles triangles that can be formed from the n-gon, and is studied in detail in the following section.

4. The isosceles triangles. The integral (1) may be written in the form

$$(7) \qquad C_1 w + C_2 = \int \frac{\frac{dz}{z^2}}{\sqrt[n]{\left[\frac{z+1}{z}\right]^{\lambda_1}\left[\frac{z-1}{z}\right]^{\lambda_2}}},$$

since $\lambda_1 + \lambda_2 + \lambda_3 = 2n$. The triangle is assumed to be isosceles, so that $\lambda_1 = \lambda_2$. Further, making the substitution

$$\frac{z^2-1}{z^2} = t^n,$$

the integral reduces to

$$(8) \qquad C_1 w + C_2 = \frac{n}{2}\int \frac{t^{n-1-\lambda_1}\, dt}{\sqrt{1-t^n}},$$

which is hyperelliptic in general.

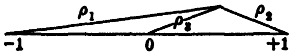
Putting $\lambda_1 = n - r$ so that $r\pi/n$ is one of the two equal angles of the triangle, and allowing C_1 and C_2 to be $n/2$ and 0 respectively, the integral becomes

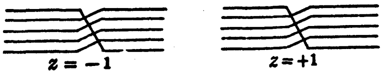
$$w = \int^t \frac{t^{r-1}\, dt}{\sqrt{1-t^n}}.$$

Hence the theorem :—*An isosceles triangle in the w-plane whose angles are $r\pi/n$, $r\pi/n$, $(n-2r)\pi/n$ is represented conformally upon the upper half of the z-plane by the equations*

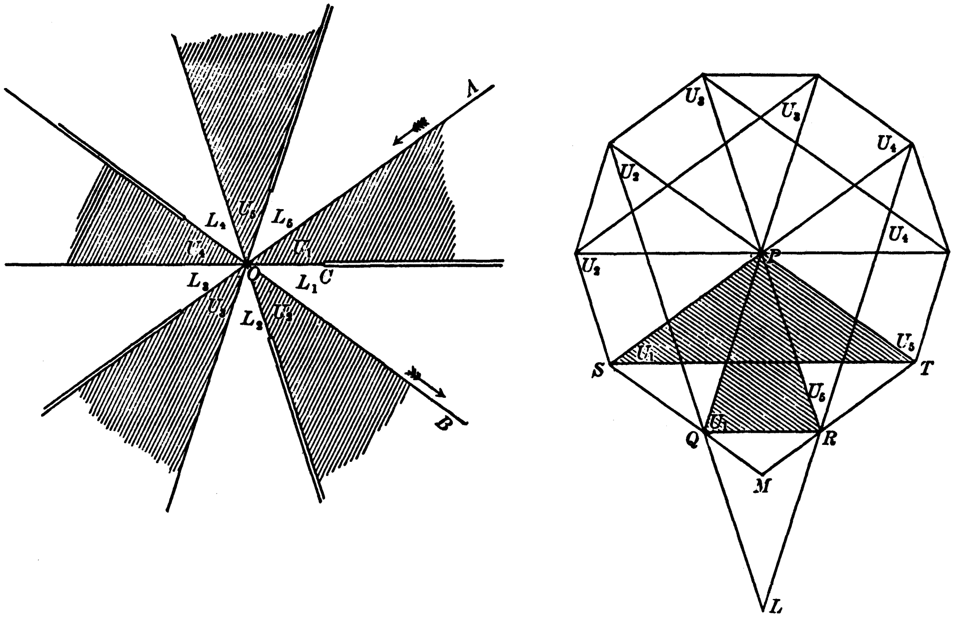
$$(9) \qquad w = \int^t \frac{t^{r-1}\, dt}{\sqrt{1-t^n}}, \quad t^n = \frac{z^2-1}{z^2}.$$

Proceeding to the geometric significance of these equations, there is, first of all, an n-leaved Riemann surface spread over the z-plane. This surface can be rendered simply-connected by a cut and proper bridging along the segment of the real axis extending from -1 through 0 to $+1$. This surface is transformed, by the substitution $\frac{z^2-1}{z^2} = t^n$, into a two-leaved Riemann surface spread over the t-plane and having winding points at the places representing the n roots of unity. Infinity is a winding point or not according as n is odd





z-plane, a Riemann surface with five leaves. $p = 2$.



t-plane, a Riemann surface with two leaves. $p = 2$.

w-plane.

FIG. 1.

or even. This surface may be made simply connected by cuts extending radially from the finite winding points to infinity. Consider, next, what corresponds to the real z-axis.

Putting

$$z + 1 = \rho_1 e^{i\theta_1}, \quad z - 1 = \rho_2 e^{i\theta_2}, \text{ and } z = \rho_3 e^{i\theta_3},$$

then

$$t = \sqrt[n]{\frac{\rho_1\rho_2}{\rho_3^2}}\, e^{i\left(\frac{\theta_1 + \theta_2 - 2\theta_3}{n}\right)}.$$

If, now, z starts at 0 and describes the real axis in the positive direction, one of the values of t will start at ∞ and describe the line inclined to the axis of reals at an angle π/n. The segment of the z-axis between 0 and 1 corresponds to the line AO [see Figure 1]. The segment of the z-axis from 1 to ∞ corresponds to the line OC from $t = 0$ to $t = 1$. Here t encounters the cut extending from 1 to ∞ and descends to the lower leaf returning to 0 as z describes the segment from ∞ to -1. Finally, as z describes the segment from -1 to 0, t describes the line OB inclined to the real axis at an angle $-\pi/n$ and extending, on the lower leaf, from 0 to ∞. The upper half of the first leaf, say, of the z-surface is thus correlated to the infinite sector AOB, one-half of which lies in the upper leaf of the t-surface and the other half in the lower leaf. The complete correlation of the two surfaces can now be effected by a simple inspection of the figure, as is indicated in Figure 1, where U_i and L_i stand for the upper and lower halves of the i^{th}-leaf, respectively.

Choosing ∞ for the lower limit of the integral in (9) so that when $t = \infty$, $w = 0$, the path of w corresponding to the t-path $AOCOB$ is found to be the given isosceles triangle situated with its vertex at the origin and its base below and parallel to the real w-axis, viz., at PQR in the figure. For when t describes the line AO, it is always equal to $e^{\frac{\pi i}{n}}x$. Making this substitution for t in the integral (9), it becomes

$$w = e^{\frac{r\pi i}{n}} \int_\infty^x \frac{x^{r-1}dx}{\sqrt{1+x^n}}.$$

The integral

$$\int_\infty^x \frac{x^{r-1}dx}{\sqrt{1+x^n}}$$

is obviously real and finite for all real positive values of x. Hence w describes a line inclined to the real w-axis at an angle $r\pi/n$.

The integrals

$$(10) \qquad \int_0^\infty \frac{x^{r-1}dx}{\sqrt{1+x^n}} \text{ and } \int_0^1 \frac{x^{r-1}dx}{\sqrt{1-x^n}},$$

for which we shall use the symbols R_r and K_r respectively, are expressible as Γ-functions and are real positive constants.

They are related, as the theory of the Γ-function shows, by the equation

$$R_r \cos \frac{r\pi}{n} = K_r. \tag{11}$$

It is easy to complete the w-path corresponding to the t-path $AOCOB$. The following table exhibits the correspondence:

$$\begin{array}{llllll} z = & 0, & 1, & \infty, & -1, & 0; \\ t = & \infty, & 0, & 1, & 0, & \infty; \\ w = & 0, & -e^{\frac{r\pi i}{n}} R_r, & -iR_r \sin \frac{r\pi}{n}, & e^{-\frac{r\pi i}{n}} R_r, & 0; \end{array}$$

w thus passes around the triangle in the positive direction. The vertices of the triangle are at 0, $-e^{\frac{r\pi i}{n}} R_r$, $e^{-\frac{r\pi i}{n}} R_r$ and its angles are $(n - 2r)\pi/n$, $r\pi/n$, $r\pi/n$, respectively.

The figure shows four other triangles corresponding to the upper halves of the four other leaves of the z-surface.

The number of linearly independent integrals of the form

$$\int \frac{t^{r-1} dt}{\sqrt{1 - t^n}}$$

is $\frac{1}{2}(n - 2)$ if n is even, and $\frac{1}{2}(n - 1)$ if n is odd. Again, the number of distinct isosceles triangles that can be formed from the regular n-gon is $\frac{1}{2}(n - 2)$ for n even, and $\frac{1}{2}(n - 1)$ for n odd. Hence there is a one to one correspondence between these triangles and the hyperelliptic integrals. This result may be embodied in the following theorem:

The totality of distinct isosceles triangles that can be formed from a regular n-gon is represented upon the upper half of the z-plane by the complete set of linearly independent hyperelliptic integrals of the form

$$\int \frac{t^{r-1} dt}{\sqrt{1 - t^n}};$$

t being connected with z by the relation

$$t^n = \frac{z^2 - 1}{z^2} \, .$$

The isosceles triangles are not the only ones that can be treated by means of hyperelliptic integrals. In fact, every right triangle whose angles are

$$\frac{\pi}{2}\, , \; \frac{r\pi}{n}\, , \; \frac{\left(\frac{n}{2} - r\right)\pi}{n}$$

can be conformally represented upon the upper half of the z-plane by means of the relations

$$w = \int \frac{t^{\frac{r-2}{2}}\, dt}{\sqrt{1 - t^{\frac{n}{2}}}}, \quad 2t^{\frac{n}{2}} = \frac{z - 1}{z} \, . \tag{12}$$

This follows upon reducing integral (7) by the substitution $2t^{\frac{n}{2}} = \frac{z-1}{z}$, properly choosing constants, and afterwards assuming

$$\frac{\lambda_1}{n} = \frac{1}{2}\, , \; \lambda_2 = n - r.$$

The fact is apparent geometrically, since every right triangle is exactly one-half the corresponding isosceles triangle.

5. Special cases. In this section it is proposed to study in more detail the triangles formed from n-gons in which n has the values, 3, 4, · · · 10, and, in particular, to determine all the triangles whose conformal representation can be effected by hyperelliptic integrals of deficiency 2. The cases $n = 3$, 4 are well known and lead to elliptic integrals. The representation is effected in these cases by the relations*

$$w = \int_{\infty}^{t} \frac{dt}{\sqrt{1 - t^3}}, \quad t^3 = \frac{z^2 - 1}{z^2}, \tag{13}$$

---

* Love: Vortex Motion in Certain Triangles, *Amer. Jour.*, vol. 11 (1889). The integral expressions and the orientation of the triangles are completely studied in this article.

and

$$w = \int_{\infty}^{t} \frac{dt}{\sqrt{1 - t^4}}, \quad t^4 = \frac{z^2 - 1}{z^2},$$

respectively.

In these cases the triangles completely cover the w-plane without overlapping.

For $n = 5$, we have $\nu = 2$. The two triangles are [4, 4, 2] and [3, 3, 4]; *i. e.*, they are both isosceles. The conformal representation is effected by the equations

$$(14) \qquad w_1 = \int_{\infty}^{t} \frac{dt}{\sqrt{1 - t^5}}, \quad w_2 = \int_{\infty}^{t} \frac{tdt}{\sqrt{1 - t^5}}, \quad \text{and } t^5 = \frac{z^2 - 1}{z^2}$$

Figure 1 exhibits the correspondence, the triangles PST and PQR corresponding to the upper half of the first leaf of the z-surface. Here the w-plane cannot be covered completely without overlapping and it is natural to seek some surface that can be so covered.

For this purpose I choose two integrals:

$$(15) \qquad w_1^1 = \frac{1}{R_1} \int_{\infty}^{t} \frac{dt}{\sqrt{1 - t^5}}, \qquad w_2^1 = \frac{1}{R_2} \int_{\infty}^{t} \frac{tdt}{\sqrt{1 - t^5}},$$

where R_1 and R_2 have the definitions given in (10). This causes the vertices S, T, and Q, R of the two triangles to lie on the unit circle; thus, if a is $e^{\frac{\pi i}{5}}$, then S and T are at a^6 and a^9, respectively, while Q and R lie at a^7 and a^8, respectively. If, now, v_1 and v_2 are two integrals connected with w_1^1 and w_2^1 by the equations

$$(16) \qquad \begin{aligned} a^6 v_1 + a^9 v_2 &= w_1^1, \\ a^7 v_1 + a^8 v_2 &= w_2^1 \end{aligned}$$

then the vertices of the two triangles will be given by the following values of v_1 and v_2:

$$P: \begin{Bmatrix} v_1 = 0 \\ v_2 = 0 \end{Bmatrix}; \; S \text{ and } Q: \begin{Bmatrix} v_1 = 1 \\ v_2 = 0 \end{Bmatrix}; \; R \text{ and } T: \begin{Bmatrix} v_1 = 0 \\ v_2 = 1 \end{Bmatrix}.$$

Constructing the triangles SMT and QLR which correspond to the lower half of the first leaf of the z-surface, it is seen that the vertices M and L are given by the values $v_1 = 1$, $v_2 = 1$. This suggests the following geometric device: if the quadrilateral $PSMT$ be bent along the line PM and the quadrilateral $PQLR$ along the line QR, the two may be fitted together to form a tetrahedron $PQRL$ (Figure 2), whose vertices correspond, respectively, to the following values:

$$P:\left\{\begin{matrix} v_1 = 0 \\ v_2 = 0 \end{matrix}\right\}; \quad Q:\left\{\begin{matrix} v_1 = 1 \\ v_2 = 0 \end{matrix}\right\}; \quad R:\left\{\begin{matrix} v_1 = 0 \\ v_2 = 1 \end{matrix}\right\}; \quad L:\left\{\begin{matrix} v_1 = 1 \\ v_2 = 1 \end{matrix}\right\}.$$

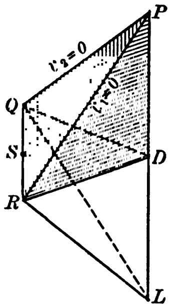


FIG. 2.

The points S and D (where S bisects QR) both correspond to the values $v_1 = v_2 = \frac{1}{2}$.

To any *given* z in the upper half of the first leaf of the Riemann surface over the z-plane there correspond in general *two* points in the shaded portion of the tetrahedron, the exceptions being for $z = 0, +1$, or -1 which correspond respectively to P, Q, R. $z = \infty$ corresponds to the two points S and D. Five such tetrahedra, connected together at one vertex, form a complete picture of the z-Riemann surface.

For $n = 6$ we have $\nu = 3$. There are two isosceles triangles, $[5, 5, 2]$ and $[4, 4, 4]$, and one right triangle $[5, 4, 3]$. The right triangle is elliptic; *i. e.*, it is represented conformally upon the upper half of the z-plane by an elliptic integral. Taken in connection with the triangles noticed for $n = 3$ and 4, respectively, it completes the set of elliptic triangles.

The conformal representation of the isosceles triangles is effected by the equations

$$(17) \qquad w_1 = \int_\infty^t \frac{dt}{\sqrt{1 - t^6}}, \quad w_2 = \int_\infty^t \frac{t\,dt}{\sqrt{1 - t^6}}, \quad t^6 = \frac{z^2 - 1}{z^2}.$$

The tetrahedron formed as in the case for $n = 5$ is regular, showing clearly the possibility of representing the triangles conformally upon the upper half-plane by means of elliptic functions.* In fact, the substitution $t^2 = 1/s$

---

* Klein, *Vorlesungen über das Ikosaeder*.

in the first integral of (16), $t^2 = s$ in the second, with proper choice of constants, reduces each of them to the same elliptic integral; viz:

$$\int \frac{dy}{\sqrt{4y^3 - 4}}.$$

The correlation of the two triangles can then be effected completely by the relations

$$(18) \qquad p^3 w = \frac{z^2}{z^2 - 1} \text{ and } p^3 w = \frac{z^2 - 1}{z^2},$$

respectively.*

The places $z = 0, 1, -1$ which correspond to the vertices of the triangles are seen to be the zeros and infinities of the elliptic function pu, defined by the equation

$$p'u = \sqrt{4p^3u - 4}.$$

The period-parallelogram is a rhombus (Figure 3). If this rhombus be doubly-covered, cut and bridged along the line AB, and the two leaves sewed together along the outer edges, it may be completely covered without overlapping by alternately shaded and unshaded triangles of either sort, [5, 5, 2] or [4, 4, 4]. The rhombus, so constructed, will then be a complete picture of the corresponding z-Riemann surface, there being just twelve equal regions by either mode of division. That such a representation is possible, is at once apparent from the fact that both isosceles triangles are the doubles of the same elliptic triangle [5, 4, 3], and thus the square-root process in the complex-plane is suggested.†

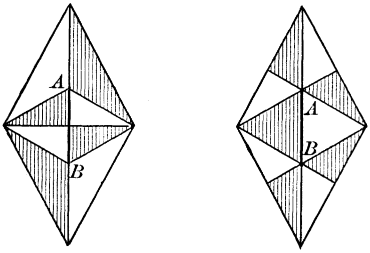


FIG. 3.

For $n = 7$, $\nu = 4$. There are three isosceles triangles which are conformally represented upon the upper half-plane by the three hyperelliptic integrals

$$\int_{\infty}^{t} \frac{t^{r-1}dt}{\sqrt{1 - t^7}} \; (r = 1, 2, 3) \text{ and } t^7 = \frac{z^2 - 1}{z^2}.$$

---

* Love, *loc. cit.*

† This representation formed the basis of a report read before the Chicago section of the American Mathematical Society, January 2nd and 3rd, 1902. See *Bulletin* for February of that year

Besides these isosceles triangles, there is a "cyclic" triangle [6, 5, 3] represented conformally upon the upper half-plane by an Abelian integral of deficiency $p = 3$. The complete set of linearly independent integrals I_k [see §2] represents this same triangle with its vertices permuted. There is no triangle formed from the heptagon which can be conformally represented upon the upper half-plane by an integral of deficiency 2.

For $n = 8$, $\nu = 5$. There are three isosceles triangles one of which is elliptic. The other two lead to integrals of deficiency 3. There is a scalene triangle [7, 6, 3], which also leads to an integral of deficiency 3. There remains the right triangle [4, 5, 7] which can be conformally represented by an integral of deficiency 2; viz., from (1),

$$C_1 w + C_2 = \int \frac{dz}{\sqrt[8]{(z+1)^4(z-1)^5 z^7}} \, .$$

The second integral of the first kind existing upon the Riemann surface is

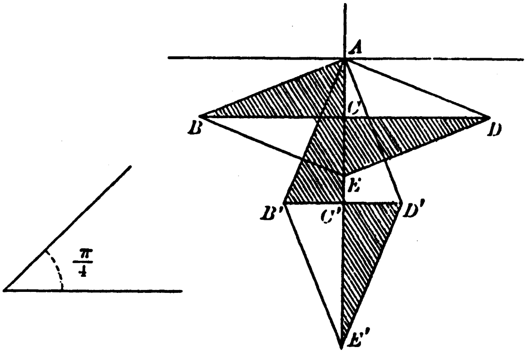
$$C_1 w + C_2 = \int \frac{dz}{\sqrt[8]{(z+1)^4(z-1)^7 z^5}} \, ,$$

which represents the same triangle with the last two vertices permuted, *i. e.*, the triangle [4, 7, 5]. These two integrals reduce to the normal hyperelliptic form by means of the substitution $2t^4 = \dfrac{z-1}{z}$, so that, by proper choice of the constants, the conformal representation is effected by the equations

$$(19) \qquad w_1 = \int_\infty^t \frac{dt}{\sqrt{t-t^5}}, \quad w_2 = \int_\infty^t \frac{t\,dt}{\sqrt{t-t^5}}, \quad 2t^4 = \frac{z-1}{z} \, .$$

These integrals also follow from formula (12).

Here we have an eight-leaved Riemann surface over the z-plane. The corresponding t-surface is two-leaved with winding points at 0, ∞, ± 1, and $\pm i$. The real axis of the first leaf, say, of the z-surface corresponds to the boundary of a sector in the first quadrant of the upper leaf of the t-surface whose vertical angle is $\pi/4$. The complete correspondence between the two surfaces is easily determined, but is not necessary for our present purpose. The w-triangles corresponding to the upper half of the first leaf of the z-surface have the positions ABC and $AB'C'$, as shown in Figure 4. By completing the parallelograms, as shown in the figure, where the unshaded triangles correspond to the lower half of the upper leaf of the z-surface,* and choosing constants properly as in the case for $n = 5$, the two parallelograms may be bent to form a tetrahedron which can be completely covered without overlapping by the triangles.



t-plane. w-plane.

FIGURE 4.

The tetrahedron can be covered by repetitions of either of the isosceles triangles, [7, 7, 2] or [5, 5, 6]. As a matter of fact each of these isosceles triangles is double the right triangle we have been discussing. The square-root process in the complex plane is again suggested. The conformal representation of the set of three isosceles triangles formed from the octagon is effected by the equations

$$(20)\quad w_1 = \int_\infty^t \frac{dt}{\sqrt{1-t^8}}, \quad w_2 = \int_\infty^t \frac{tdt}{\sqrt{1-t^8}}, \quad w_3 = \int_\infty^t \frac{t^2dt}{\sqrt{1-t^8}}, \quad t^8 = \frac{z^2-1}{z^2}.$$

The substitution $t^2 = s$ reduces the first and third of these integrals to the form given in (19), and the second to an elliptic integral. This result is perfectly analogous to that of the triangle [5, 5, 2] treated by Schwarz† and Love,† and discussed here in the case for $n = 6$.

---

*The triangles CED and $C'E'D'$ correspond to the upper half of the fifth leaf of the z-surface.

†*Loc. cit.*

For $n = 9$, $\nu = 7$. Not any of these seven triangles can be represented conformally upon the upper half of the z-plane by integrals of deficiency 2. The triangle [6, 6, 6] is elliptic, all the others correspond to integrals of deficiency either 3 or 4.

For $n = 10$, $\nu = 8$. There are four isosceles triangles and two right triangles. The two scalene triangles [9, 8, 3] and [9, 7, 4] have associated with them the triangles [8, 6, 6] and [4, 8, 8] and correspond, respectively, to integrals of deficiency 4.

From (12) the conformal representation of the right triangles [7] and [5, 6, 9] is effected by the equations

$$(21) \qquad w_1 = \int_\infty^t \frac{dt}{\sqrt{1 - t^5}}, \quad w_2 = \int_\infty^t \frac{tdt}{\sqrt{1 - t^5}}, \quad 2t^5 = \frac{z - 1}{z}.$$

These integrals are, of course, exactly the same as in the case for $n = 5$, since the right triangles are, respectively, the halves of the isosceles triangles there considered. The same tetrahedron will serve for the surface which is to be simply covered.

The four isosceles triangles are [9, 9, 2], [8, 8, 4], [7, 7, 6], and [6, 6, 8]. The second and fourth of these are the ones considered in the case for $n = 5$. The first and third are also double the right triangles [5, 6, 9] and [5, 8, 7], respectively. The conformal representation of the set of four isosceles triangles is effected by the equations

$$(22) \qquad w_r = \int_\infty^t \frac{t^{r-1}dt}{\sqrt{1 - t^{10}}}, \quad t^{10} = \frac{z^2 - 1}{z^2}; \quad (r = 1, 2, 3, 4.)$$

The second and fourth of these integrals reduce to the form (21) by the substitution $t^2 = s$, while the first and third reduce to the same form by the substitution $t^2 = 1/s$. Thus we have a second example analogous to the triangle [5, 5, 2]. The same tetrahedron [Figure 2] can be simply covered by the triangles [9, 9, 2] and [7, 7, 6].

We have now completed our search for triangles corresponding to integrals of deficiency 2, for it is evident from formulas (9) and (12) that n cannot be greater than 12. On examination we find that not one of the twelve possible triangles that can be formed from the dodecagon, corresponds to an integral of deficiency 2 [or less] except those already discussed, as occurring in n-gons of fewer than twelve sides.

Bringing these triangles together, they are:

1. $[5, 5, 2]$, whose angles are $\pi/6$, $\pi/6$, $2\pi/3$.
2. $[4, 4, 2]$, whose angles are $\pi/5$, $\pi/5$, $3\pi/5$.
3. $[3, 3, 4]$, whose angles are $2\pi/5$, $2\pi/5$, $\pi/5$.
4. $[4, 5, 7]$, whose angles are $\pi/2$, $3\pi/8$, $\pi/8$.
5. $[5, 8, 7]$, whose angles are $\pi/2$, $\pi/5$, $3\pi/10$.
6. $[5, 6, 9]$, whose angles are $\pi/2$, $2\pi/5$, $\pi/10$.

Besides these triangles there are four others which can be reduced to the hyperelliptic case $p = 2$; viz:

1. $[7, 7, 2]$, whose angles are $\pi/8$, $\pi/8$, $3\pi/4$.
2. $[5, 5. 6]$, whose angles are $3\pi/8$, $3\pi/8$, $\pi/4$.
3. $[9, 9, 2]$, whose angles are $\pi/10$, $\pi/10$, $4\pi/5$.
4. $[7, 7, 6]$, whose angles are $3\pi/10$, $3\pi/10$, $2\pi/5$.

UNIVERSITY OF WISCONSIN,
AUGUST, 1903.